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Shocks in the asymmetry exclusion model with an impurity

K Mallick

Laboratoire de Mathématiques UA 762 du CNRS, DMI Ecole Normale Supérieure, 45 rue D'Ulm, F-75231 Paris 05 Cedex, France
and
Service de Physique Théorique, CE Saclay, F-91191 Gif-sur-Yvette Cedex, France

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Abstract. We consider the one-dimensional asymmetric exclusion process with an impurity. This model describes particles hopping in one direction with stochastic dynamics and a hard core exclusion condition. The impurity hops with a rate different from that of the normal particles and can be overtaken by these particles. We solve this model exactly and give its phase diagram. In one of the phases the system presents a shock, i.e. a sharp discontinuity between a region of high density of particles and a region of low density. Density profiles and relevant exponents are explicitly calculated. These exact results for systems of finite size are consistent with anomalous diffusion laws observed in infinite systems.

1. Introduction

The one-dimensional asymmetric simple-exclusion process (ASEP) is a model of particles hopping in a preferred direction with stochastic dynamics and hard core exclusion. A model of this kind usefully describes many different physical phenomena such as hopping conductivity, diffusion of particles through narrow pores, growth processes and traffic flows [1–4]. The mathematical relevance of the ASEP is that it is a discrete version of the Burgers equation in the appropriate scaling limit, and a lot of work has been devoted to the derivation of the hydrodynamical limit from microscopic dynamics [5, 6]. It is well known that the Burgers equation exhibits shocks. The question of the existence and description of these shock structures at the microscopic level has been studied both numerically and analytically [7–15], and several models have been proposed. In [14], the authors calculated the shape of a shock as seen from a second class particle. These second class particles overtake holes but not normal particles, while normal (or first class) particles overtake both holes and second class particles. A more direct way to provoke a shock [10] is to introduce a 'slow' link in the system: the particles can cross this link with rate r, which is less than the crossing rate of all the other bonds (taken equal to 1). A mean-field analysis, supported by numerical simulations, shows that a shock appears on the macroscopic level, when r is less than a critical value r_c (function of density). The authors measured the width of the shock as a function of the size L of the system and showed that it scales as $L^{1/3}$ or $L^{1/2}$ depending upon whether particle–hole symmetry exists or not (i.e. whether the density of particles is equal to $\frac{1}{2}$ or not). This model has not yet been solved exactly.

In this paper, we study the ASEP on a ring with one defective particle (say an impurity) that jumps with a rate α ($\leqslant 1$) less than that of other particles and can be overtaken with rate β ($\leqslant 1$) by normal particles (this model has been introduced in [7] and [16]).

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We use the matrix technique introduced in [14, 17, 18] for solving exactly certain exclusion processes. This enables us to derive analytic formulae for the current in the system and for the density profile (defined as the mean occupation of a site in the stationary state). In the thermodynamic limit the system can exist in different phases in the (α, β) plane. We determine the relevant correlation lengths and the critical exponents that characterize the divergence of these correlation lengths on the lines bounding the different phases.

With this exact solution, we show that there exists one phase of the system in which the impurity causes a shock, i.e. a spatial segregation between a region of high density of particles and a region of low density. Because of the fluctuations of the position of the impurity, the width of the shock always scales as $L^{1/2}$ whether the particle–hole symmetry exists or not.

This paper is organized as follows. In section 2 we describe the model and write the weights of configurations in the stationary state as a trace of an infinite-dimensional matrix. In section 3 we obtain exact results for finite systems and demonstrate the appearance of a shock in the thermodynamic limit. Section 4 is devoted to the calculation of density profiles and concluding remarks are given in the last section.

2. The model. Expression of the stationary state with a matrix ansatz

We shall consider the totally asymmetric exclusion process on a lattice of $L + 1$ sites; the sites are numbered from 0 to L. There are N 'normal' particles ($0 \leqslant N \leqslant L$), denoted by 1, in the system, and one impurity, denoted by 2. Each site i is either occupied by one normal particle, or by the impurity, or is empty. The lattice has periodic boundary conditions: site i is the same as site $i + L + 1$.

Stochastic dynamical rules govern the evolution of the system. During the infinitesimal time step $\mathrm{d}t$, any bond $(i, i + 1)$ (with $0 \leqslant i \leqslant L$) evolves as follows:

$$\begin{aligned} &10 \rightarrow 01 \text{ with rate } 1 \\ &20 \rightarrow 02 \text{ with rate } \alpha \\ &12 \rightarrow 21 \text{ with rate } \beta \end{aligned} \tag{1}$$

All other transitions are forbidden.

In the particular case where $\alpha = 1$ and $\beta = 0$ the impurity is identical to a normal particle; it can be considered as a tracer or a 'tagged' particle. The fluctuations of the position of a tracer have been studied in [18]. When $\alpha = 1$ and $\beta = 1$, the impurity is a second class particle which has been used in the study of shock fluctuations in [14].

We shall work in the relative frame of the impurity unless the contrary is specified, i.e. we use the translation invariance of the system to relabel the sites, so that the impurity always remains on site number 0.

The system has $\binom{L}{N} = \frac{L!}{N!(L-N)!}$ configurations. In the long time limit, the system reaches a stationary state in which each configuration $\mathcal{C}$ has a stationary probability $p(\mathcal{C})$. The computation of the $p(\mathcal{C})$'s is non-trivial; they can be expressed [14, 16] as a trace of a product of non-commuting operators D, E and A:

$$p(\mathcal{C}) = \frac{1}{Z_{L,N}} \operatorname{Tr}\left(A \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) \right) \tag{2}$$

where $\tau_i = 1$ if site i is occupied by a particle in configuration $\mathcal{C}$ and $\tau_i = 0$ if it is empty.

The matrices D, E and A satisfy the following algebra:

$$\begin{aligned} DE &= D + E \\ DA &= \frac{1}{\beta} A \\ AE &= \frac{1}{\alpha} A. \end{aligned} \tag{3}$$

The normalization factor $Z_{L,N}$ ensures that $\sum_{\mathcal{C}} p(\mathcal{C}) = 1$ and can be written as a trace:

$$Z_{L,N} = \mathrm{Tr}(AG_{L,N}) \tag{4}$$

with

$$G_{L,N} = \sum_{\{\tau_i=0,1\}} \delta\left(N - \sum_{i=1}^{L} \tau_i\right) \prod_{i=1}^{L} (\tau_i D + (1-\tau_i)E). \tag{5}$$

Here $\delta(x)$ is the Kronecker $\delta_{x,0}$; hence the matrix $G_{L,N}$ is the sum of all matrices formed by multiplying N D's and $(L-N)$ E's in all possible orders. The fact that the weights $p(\mathcal{C})$ given by (2) are stationary can be proved by using the same arguments as given in [14, 17, 19]. Therefore we shall not give the proof here. As shown in [17] the matrices D and E have to be infinite-dimensional unless $\alpha + \beta = 1$. A useful representation of algebra (3) is the following:

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & . & . \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & . & \\ . & & & & . & . \\ . & & & & & . \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . \\ 1 & 1 & 0 & 0 & & \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ . & & & . & . & \\ . & & & & . & . \end{pmatrix} \qquad A = |V\rangle\langle W| \tag{6}$$

where

$$\langle W| = \kappa \left(1, \quad \left(\frac{1-\alpha}{\alpha}\right), \quad \left(\frac{1-\alpha}{\alpha}\right)^2 \quad . \quad . \right) \qquad \text{and} \qquad |V\rangle = \kappa \begin{pmatrix} 1 \\ \left(\frac{1-\beta}{\beta}\right) \\ \left(\frac{1-\beta}{\beta}\right)^2 \\ . \\ . \end{pmatrix}.$$

To ensure that $\langle W|V\rangle = 1$, we take $\kappa^2 = (\alpha + \beta - 1)/\alpha\beta$. We remark that $|V\rangle$ is a right eigenvector of D and that $\langle W|$ is a left eigenvector of E:

$$D|V\rangle = \frac{1}{\beta}|V\rangle \tag{7}$$

$$\langle W|E = \frac{1}{\alpha}\langle W|. \tag{8}$$

3. Exact results for the current of first class particles and for the speed of the impurity

The fundamental quantity to calculate is the normalization factor $Z_{L,N} = \mathrm{Tr}(AG_{L,N}) = \langle W|G_{L,N}|V\rangle$ which plays a role analogous to the partition function in equilibrium statistical mechanics. The following exact formula is derived in appendix A:

$$Z_{L,N} = \frac{\alpha\beta}{(1-\alpha)(1-\beta)}\binom{L}{N}\left(\sum_{p=0}^{\infty}\binom{L}{N+p}\left(\frac{1-\alpha}{\alpha}\right)^p + \sum_{q=1}^{\infty}\binom{L}{L-N+q}\left(\frac{1-\beta}{\beta}\right)^q\right)$$
$$+\frac{1-\alpha-\beta}{(1-\alpha)(1-\beta)}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\binom{L}{N+p}\binom{L}{L-N+q}\left(\frac{1-\alpha}{\alpha}\right)^p\left(\frac{1-\beta}{\beta}\right)^q. \tag{9}$$

From this exact formula the asymptotic behaviour of $Z_{L,N}$ in the thermodynamic limit ($L \to \infty$, $N \to \infty$, the density $\rho = \frac{N}{L+1}$ being constant) can be extracted. Different forms are obtained in different ranges of α, β and ρ:

- $\rho < \beta < 1-\alpha$ and $\rho < 1-\alpha < \beta$

$$Z_{L,N} \simeq \frac{\beta(1-\alpha-\rho)}{(1-\alpha)(\beta-\rho)}\frac{\binom{L}{N}}{\alpha^{L-N}(1-\alpha)^N} \tag{10}$$

- $\beta < \rho < 1-\alpha$

$$Z_{L,N} \simeq \frac{(1-\alpha-\beta)}{(1-\alpha)(1-\beta)}\frac{1}{\alpha^{L-N}(1-\alpha)^N\beta^N(1-\beta)^{L-N}} \tag{11}$$

- $1-\alpha < \rho < \beta$

$$Z_{L,N} \simeq \frac{\alpha\beta(\alpha+\beta-1)\rho(1-\rho)}{(\beta-\rho)^2(\alpha+\rho-1)^2}\frac{\binom{L}{N}^2}{L} \tag{12}$$

- $\beta < 1-\alpha < \rho$ and $1-\alpha < \beta < \rho$

$$Z_{L,N} \simeq \frac{\alpha(\rho-\beta)}{(1-\beta)(\rho+\alpha-1)}\frac{\binom{L}{N}}{\beta^N(1-\beta)^{L-N}}. \tag{13}$$

The process is invariant when the particles and holes are interchanged, the direction of motion reversed and α and β exchanged (charge conjugation and reflection symmetry [20]). Analytically, the following symmetry is thus satisfied:

$$\begin{aligned} &N \to L-N \\ &\rho \to 1-\rho \\ &\alpha \to \beta \\ &\text{site number } i \to \text{site number } L+1-i. \end{aligned} \tag{14}$$

This fact can be checked on equations (10) to (13).

In the stationary state, the speed $\boldsymbol{V}$ of the impurity in the reference frame of the lattice can be expressed with the help of algebra (3) as follows:

$$\begin{aligned} \boldsymbol{V} &= \alpha\boldsymbol{E}(\tau_{i+1}=0|\tau_i=2) - \beta\boldsymbol{E}(\tau_{i-1}=1|\tau_i=2) \\ &= \frac{\alpha\langle W|EG_{L-1,N}|V\rangle - \beta\langle W|G_{L-1,N-1}D|V\rangle}{Z_{L,N}} \\ &= \frac{Z_{L-1,N}-Z_{L-1,N-1}}{Z_{L,N}}. \end{aligned} \tag{15}$$

Here $\boldsymbol{E}$ is the expectation in the stationary state, and $\boldsymbol{E}(|)$ the conditional expectation. Similarly the current of the first class particles in the frame of the lattice is

$$J = \rho\frac{Z_{L-1,N}}{Z_{L,N}} + (1-\rho)\frac{Z_{L-1,N-1}}{Z_{L,N}} = \rho\boldsymbol{V} + \frac{Z_{L-1,N-1}}{Z_{L,N}}. \tag{16}$$

The second term on the RHS of (16) has a direct interpretation: it is the current of first class particles in the relative frame of the impurity. The proof of (16) is given in appendix B.

With the help of the asymptotic expressions for $Z_{L,N}$ (10)–(13), we obtain the following formulae for the current J and the speed $\boldsymbol{V}$:

- for $\rho < \beta < 1-\alpha$ and $\rho < 1-\alpha < \beta$: $J = \rho(1-\rho)$ and $\boldsymbol{V} = \alpha - \rho$
- for $\beta < \rho < 1-\alpha$: $J = \rho(\alpha-\beta) + \beta(1-\alpha)$ and $\boldsymbol{V} = \alpha - \beta$
- for $1-\alpha < \rho < \beta$: $J = \rho(1-\rho)$ and $\boldsymbol{V} = 1 - 2\rho$
- for $\beta < 1-\alpha < \rho$ and $1-\alpha < \beta < \rho$: $J = \rho(1-\rho)$ and $\boldsymbol{V} = 1 - \beta - \rho$

The results for the case $\beta < 1-\alpha$ were previously obtained in [16] where a different expression was used for $Z_{L,N}$. The derivatives of the current and the speed are discontinuous at the boundaries of the domains that appear above; therefore, following [21], the phase transition can be considered to be of first order. In the phase $\beta < \rho < 1-\alpha$, the speed of the impurity does not depend upon the density ρ of the particles [16]: this suggests that the impurity moves as if the site ahead of it was always empty and the site behind it always occupied. In the next section, by calculating explicitly the density profile in the whole range of α and β, we show that this phase presents a density shock: the particles pile up behind the impurity.

4. Computation of the density profile

We denote by n_i the expectation that site i is occupied in the stationary state, knowing that the impurity is at site zero. With the help of the algebra (3), it is straightforward to write the following identity for n_i:

$$n_i = \boldsymbol{E}(\tau_i = 1 | \tau_0 = 2) = \sum_{p=0}^{N-1} \frac{\langle W | G_{i-1,p} D G_{L-i,N-p-1} | V \rangle}{Z_{L,N}}. \tag{17}$$

Here we have the convention that $G_{k,q} = 0$ whenever $q < 0$ or $q > k$. It is not easy to calculate n_i from (17); however, its discrete derivative can be brought to a much simpler form:

$$n_i - n_{i+1} = \frac{1}{Z_{L,N}} \langle W | \sum_{p=0}^{N-1} G_{i-1,p} D G_{L-i,N-p-1} - G_{i,p} D G_{L-i-1,N-p-1} | V \rangle. \tag{18}$$

We substitute the general identity $G_{j,k} = G_{j-1,k-1} D + G_{j-1,k} E$ in equation (18):

$$\begin{aligned}
\sum_{p=0}^{N-1} & G_{i-1,p} D G_{L-i,N-p-1} - G_{i,p} D G_{L-i-1,N-p-1} \\
&= \sum_{p=0}^{N-1} G_{i-1,p} D (D G_{L-i-1,N-p-2} + E G_{L-i-1,N-p-1}) \\
&\quad - (G_{i-1,p-1} D + G_{i-1,p} E) D G_{L-i-1,N-p-1}.
\end{aligned}$$

The first and third terms cancel each other. The explicit representation (6) provides that $DE - ED = |1\rangle\langle 1|$. We thus obtain

$$n_i - n_{i+1} = \sum_{p=0}^{N-1} \frac{\langle W | G_{i-1,p} | 1 \rangle \langle 1 | G_{L-i-1,N-p-1} | V \rangle}{Z_{L,N}}. \tag{19}$$

The values of n_1 and n_L can be determined directly:

$$n_1 = \frac{\mathrm{Tr}(ADG_{L-1,N-1})}{Z_{L,N}} = 1 - \frac{\mathrm{Tr}(AEG_{L-1,N})}{Z_{L,N}} = 1 - \frac{1}{\alpha} \frac{Z_{L-1,N}}{Z_{L,N}} \tag{20}$$

$$n_L = \frac{\mathrm{Tr}(DAG_{L-1,N-1})}{Z_{L,N}} = \frac{1}{\beta}\frac{Z_{L-1,N-1}}{Z_{L,N}}. \tag{21}$$

Expressions (19) and (20) or (21) allow a full knowledge of the density profile $n_i (i = 1, \ldots, L)$ for any finite system and can be used for exact numerical computations.

We calculated the density profile explicitly in the thermodynamic limit with the help of the asymptotic expressions of $Z_{L,N}$ given in (10)–(13). These asymptotic expressions were substituted in formula (19). After carefully determining the term that dominates the sum in different sectors of α, β and ρ, the saddle point method is applied and the density profile is obtained. We thus derived the phase diagram of the system given in figure 1. There are six distinct phases. By using the symmetry (14), it is enough to describe precisely only four out of the six phases.

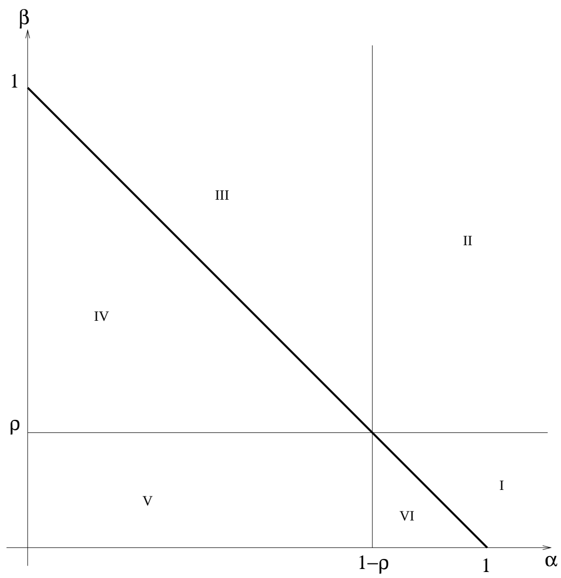

Figure 1. The phase diagram of the system.

The phase diagram is divided into two domains by the line $\beta = 1 - \alpha$. On this line the matrices D, E and A can be chosen as scalars: $D = \frac{1}{\beta}$, $E = \frac{1}{\alpha}$, $A = 1$; each configuration then has the same stationary probability, $p(\mathcal{C}) = 1/\binom{L}{N}$ and the density profile is flat, i.e. $n_i = \rho$ for all i.

In the domain $\beta > 1 - \alpha$ (figure 2), the impurity creates a local disturbance in the system and the density profile in the bulk is flat in the thermodynamic limit. It is possible to distinguish three phases in this domain.

4.1. Phase I. $1 - \alpha < \beta < \rho$

In this case, $n_1 = 1 - \frac{1-\beta}{\alpha}(1 - \rho)$ and $n_L = \rho$; the density profile decreases exponentially from n_1 to its bulk value ρ; the disturbance due to the impurity has a characteristic length ξ such that in the intermediate scale $1 \ll i \ll L$:

• when $\frac{\beta\rho}{(1-\beta)(1-\rho)} < (\frac{1-\alpha}{\alpha})^2$, $(n_i - \rho) \sim \exp(-i/\xi)$ with $\xi^{-1} = -\ln(1 - (\rho - 1 + \alpha)\frac{1-\alpha-\beta}{\alpha(1-\alpha)})$.

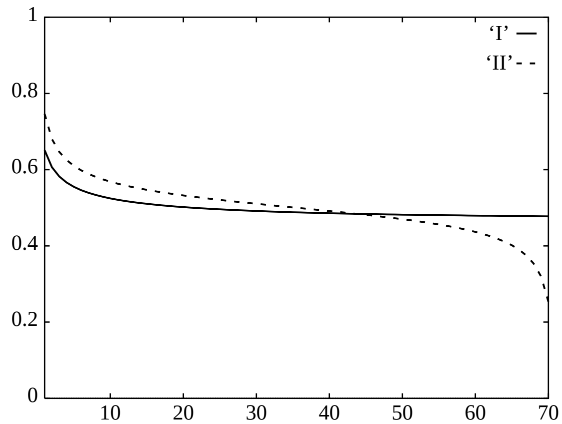

Figure 2. Typical density profiles in the $\beta > 1 - \alpha$ sector with $L = 70$ and $\rho = 0.5$. The curves represent the probability (vertical axis) that a site i (horizontal axis) is occupied in the stationary state. For curve I in phase I, $\alpha = 0.9$ and $\beta = 0.4$. For curve II in phase II, $\alpha = 1$ and $\beta = 1$.

- When $\frac{\beta\rho}{(1-\beta)(1-\rho)} > (\frac{1-\alpha}{\alpha})^2$, $(n_i - \rho) \sim \frac{1}{i^{1/2}} \exp(-i/\xi)$ with $\xi^{-1} = -2\ln(\sqrt{\beta\rho} + \sqrt{(1-\beta)(1-\rho)})$. In this case, we note that ξ diverges as $\frac{4\beta(1-\beta)}{(\rho-\beta)^2}$ when $\rho \to \beta$.

4.2. Phase II. $1 - \alpha < \rho < \beta$

Here $n_1 = 1 - \frac{1}{\alpha}(1-\rho)^2 > n_L = \frac{\rho^2}{\beta}$. There is no longer a characteristic length that measures the size of the disturbance in the vicinity of the impurity. The density profile reaches its bulk value ρ algebraically like $i^{-1/2}$ at a distance i from the impurity; for $1 \ll i \ll L$, we have

$$n_i - n_{i+1} = n_{L-i+1} - n_{L-i} \simeq \sqrt{\frac{\rho(1-\rho)}{4\pi}} \frac{1}{i^{3/2}}. \tag{22}$$

This formula agrees with the one derived in [14] for the special case $\alpha = \beta = 1$.

4.3. Phase III. $\rho < 1 - \alpha < \beta$

By (14), this phase is symmetric to phase I.

In the domain $\beta < 1 - \alpha$ (figure 3), the impurity plays the role of a moving obstacle that can affect the system on a global scale by provoking a discontinuity in the density profile of first class particles.

4.4. Phase IV. $\rho < \beta < 1 - \alpha$

Here $n_1 = \rho$ is less than $n_L = \rho\frac{1-\alpha}{\beta}$, and the characteristic length measuring the range of the effect of the impurity is given by $\xi^{-1} = -\ln(1 - (\beta - \rho)\frac{1-\beta-\alpha}{\beta(1-\beta)})$. That correlation length diverges as $\frac{\beta(1-\beta)}{(\beta-\rho)(1-\alpha-\beta)}$ when $\rho \to \beta$.

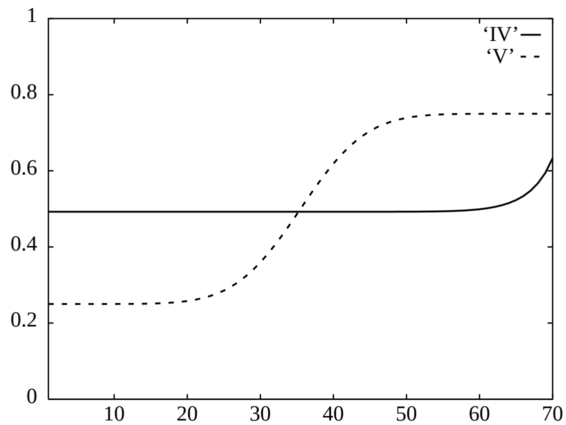

Figure 3. Density profiles in the $\beta < 1 - \alpha$ sector with $L = 70$ and $\rho = 0.5$. For curve IV in phase IV, $\alpha = 0.1$ and $\beta = 0.7$. For curve V in phase V (the shock), $\alpha = 0.25$ and $\beta = 0.25$.

4.5. Phase V. $\beta < \rho < 1 - \alpha$

In this case, we have $n_1 = \beta$ and $n_L = 1 - \alpha$. The presence of the unique impurity is enough to provoke a macroscopic shock in the system; a low density region $\rho_l = \beta$ ranging from site 1 to site $x_0 L$ is separated by a sharp interface from a region of high density $\rho_h = 1 - \alpha$ ranging from site $x_0 L$ to site L. The position of the shock x_0 is determined by $\rho = \beta x_0 + (1 - \alpha)(1 - x_0)$. When L is large, the following expression can be derived in terms of the rescaled variable $x = i/L$:

$$\frac{\mathrm{d}n(x)}{\mathrm{d}x} \simeq \sqrt{\frac{L}{2\pi(\rho(\alpha - \beta) + \beta(1 - \alpha))}}(1 - \alpha - \beta)^2 \exp\left(-\frac{L(1 - \alpha - \beta)^2 (x - x_0)^2}{2(\rho(\alpha - \beta) + \beta(1 - \alpha))}\right). \tag{23}$$

Asymptotically, $n(x)$ is an error function interpolating between the regions of low and high density with width scaling as $L^{1/2}$.

4.6. Phase VI. $\beta < 1 - \alpha < \rho$.

By (14), this phase is symmetric to phase IV.

5. Discussion and conclusion

The exact results obtained above for a finite system can be related, through a finite-size scaling analysis, to results given in [9, 12] for infinite systems. In these articles, the authors consider exclusion processes on the infinite line with initial shock configurations corresponding to an average density ρ_- (ρ_+) on the left (right) of the origin; they track the shock with the help of a second class particle. The fluctuations of the position of that second class particle can be identified with the fluctuations of the shock [22]. For a system of finite size L, let us denote by $\sigma(L, t)$ the width of the shock measured at time t and averaged over all possible realizations of the dynamics of the system from initial time 0 to time t. Finite-size scaling and equation (23) allow us to write

$$\sigma^2(L,t) = Lf\left(\frac{t}{L^z}\right) \tag{24}$$

with the scaling function f becoming a non-zero constant when its argument goes to infinity (i.e. when t goes to infinity with L large but fixed). The dynamic exponent z will be taken equal to $\frac{3}{2}$ (as shown in [23, 24], this is the value of the dynamic exponent for the ASEP without impurity). If we take the limit $L \to \infty$ first, with large but fixed t, we see that in order to have a meaningful expression, $f(x)$ has to behave like $x^{2/3}$ for small values of x. Thus for an infinite system, $\sigma^2(t) \sim t^{2/3}$.

In the case $\alpha = 1 - \beta$ (corresponding to $\rho_- = \rho_+$ in [9, 12]) there is no shock in the system but the fluctuations of the position of the second class particle can still be measured. As the density is uniform throughout the system, this particle can be anywhere and therefore $\sigma^2(L,t) = L^2 g(\frac{t}{L^z})$. The same argument as above provides that $\sigma^2(t)$ behaves like $t^{4/3}$ in the infinite system.

It would be interesting to investigate time-dependent properties of the model (for example, the behaviour of the first non-zero eigenvalue of the transition matrix in the (α, β) plane). We studied how the impurity diffuses in the system: if Y_t denotes the distance forward travelled by the impurity at time t (number of hops forward minus the number of hops backwards), one can prove [18, 25] that for large t, the variance of Y_t is linear in t, the proportionality constant $\Delta(\alpha, \beta)$ being the diffusion constant. For $\alpha = 1$ and $\beta = 0$ it is known [18] that $\Delta \sim L^{-1/2}$ (subdiffusive behaviour), whereas for $\alpha = 1$ and $\beta = 1$, Δ scales like $L^{1/2}$ (superdiffusive behaviour [25]). A precise investigation of this transition is under progress; the matrix method can be used to compute $\Delta(\alpha, \beta)$.

The relation between the matrix technique and Bethe ansatz is an open problem. For this model, it seems possible to write down the Bethe equations explicitly; using them to characterize the steady state may help to answer this question.

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Appendix A. Formula for $Z_{L,N}$

From equation (4) and from the explicit representation (6), we can write

$$|V\rangle = \kappa \sum_{q=1}^{\infty} \left(\frac{1-\beta}{\beta}\right)^{q-1} |q\rangle \qquad \text{and} \qquad \langle W| = \kappa \sum_{p=1}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^{p-1} \langle p| \tag{A1}$$

where $|q\rangle$ is the qth vector of the basis in representation (6) and $\langle p|$ is the pth vector of the dual basis in the same representation.

Therefore we have the following identity:

$$\begin{aligned} Z_{L,N} = \langle W|G_{L,N}|V\rangle &= \kappa^2 \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^{p-1} \left(\frac{1-\beta}{\beta}\right)^{q-1} \langle p|G_{L,N}|q\rangle \\ &= \frac{\alpha+\beta-1}{(1-\alpha)(1-\beta)} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^{p} \left(\frac{1-\beta}{\beta}\right)^{q} \langle p|G_{L,N}|q\rangle. \end{aligned} \tag{A2}$$

It can be directly checked by induction that

$$\langle p|G_{L,N}|q\rangle = \binom{L}{N}\binom{L}{N+p-q} - \binom{L}{N+p}\binom{L}{L-N+q} \tag{A3}$$

satisfies the following recursion:

$$\begin{aligned}\langle p|G_{L,N}|q\rangle &= \langle p|G_{L-1,N-1}D|q\rangle + \langle p|G_{L-1,N}E|q\rangle \\ &= \langle p|G_{L-1,N-1}|q-1\rangle + \langle p|G_{L-1,N-1}|q\rangle + \langle p|G_{L-1,N}|q\rangle \\ &\quad + \langle p|G_{L-1,N}|q+1\rangle \end{aligned} \tag{A4}$$

with the condition that $\langle p|G_{L,N}|q\rangle$ vanishes whenever p or q is zero. Here we have adopted the convention that the binomial coefficient $\binom{L}{M}$ is zero whenever $M < 0$ or $M > L$.

Substituting the expression of $\langle p|G_{L,N}|q\rangle$ into (A2), we obtain

$$\begin{aligned} Z_{L,N} = \frac{\alpha+\beta-1}{(1-\alpha)(1-\beta)}\Bigg[&\binom{L}{N}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{1-\alpha}{\alpha}\right)^p\left(\frac{1-\beta}{\beta}\right)^q\binom{L}{N+p-q} \\ &-\sum_{p=0}^{\infty}\binom{L}{N+p}\left(\frac{1-\alpha}{\alpha}\right)^p\sum_{q=0}^{\infty}\left(\frac{1-\beta}{\beta}\right)^q\binom{L}{L-N+q}\Bigg]. \end{aligned} \tag{A5}$$

The first term on the RHS of (A5) is modified as follows:

$$\begin{aligned} &\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{1-\alpha}{\alpha}\right)^p\left(\frac{1-\beta}{\beta}\right)^q\binom{L}{N+p-q} \\ &\quad= \sum_{r=-\infty}^{\infty}\sum_{p\geqslant\max(0,r)}^{\infty}\left(\frac{1-\alpha}{\alpha}\right)^p\left(\frac{1-\beta}{\beta}\right)^{p-r}\binom{L}{N+r} \\ &\quad= \sum_{r=-\infty}^{-1}\sum_{p=0}^{\infty}\binom{L}{N+r}\left(\frac{1-\alpha}{\alpha}\right)^p\left(\frac{1-\beta}{\beta}\right)^{p-r} \\ &\qquad+\sum_{r=0}^{\infty}\sum_{p=r}^{\infty}\binom{L}{N+r}\left(\frac{1-\alpha}{\alpha}\right)^p\left(\frac{1-\beta}{\beta}\right)^{p-r} \\ &\quad= \frac{\alpha\beta}{\alpha+\beta-1}\left(\sum_{r=-\infty}^{-1}\binom{L}{N+r}\left(\frac{1-\beta}{\beta}\right)^{-r} + \sum_{r=0}^{\infty}\binom{L}{N+r}\left(\frac{1-\alpha}{\alpha}\right)^r\right) \\ &\quad= \frac{\alpha\beta}{\alpha+\beta-1}\left[\sum_{r=0}^{\infty}\binom{L}{N+r}\left(\frac{1-\alpha}{\alpha}\right)^r + \sum_{r=1}^{\infty}\binom{L}{L-N+r}\left(\frac{1-\beta}{\beta}\right)^r\right]. \end{aligned} \tag{A6}$$

Substituting in (A5) leads to formula (9) of section 3.

Asymptotic expressions (10)–(13) are derived by making a saddle-point expansion in each sum that appears in formula (9) for $Z_{L,N}$. For instance one notices that the series $\sum_{p=0}^{\infty}\binom{L}{N+p}(\frac{1-\alpha}{\alpha})^p$ is dominated by the $p = 0$ term if $1-\alpha < \rho$ and by the $p = N(1-\alpha-\rho)$ term if $1-\alpha > \rho$. Similar considerations apply to the other two sums.

Appendix B. Expression for the current of first class particles

In the steady state the current across any bond $(i, i+1)$ of the lattice is constant:

$$J = J^{i,i+1} = \boldsymbol{E}(\tau_i = 1, \tau_{i+1} = 0) + \beta\boldsymbol{E}(\tau_i = 1, \tau_{i+1} = 2). \tag{B1}$$

In the stationary state, the impurity can be on any of the $L+1$ sites with the same probability (the system is translation-invariant), so we have

$$\boldsymbol{E}(\tau_i = 1, \tau_{i+1} = 2) = \boldsymbol{E}(\tau_i = 1|\tau_{i+1} = 2)\boldsymbol{E}(\tau_{i+1} = 2) = \frac{1}{\beta}\frac{Z_{L-1,N-1}}{Z_{L,N}}\frac{1}{L+1}. \tag{B2}$$

We transform the first term in the RHS of (B1) by conditioning on the position of the impurity:

$$\begin{aligned}\boldsymbol{E}(\tau_i = 1, \tau_{i+1} = 0) &= \sum_{k=0;k\neq i,i+1}^{L} \boldsymbol{E}(\tau_i = 1, \tau_{i+1} = 0|\tau_k = 2)\boldsymbol{E}(\tau_k = 2) \\ &= \frac{1}{L+1}\sum_{j=1}^{L-1}\boldsymbol{E}(\tau_j = 1, \tau_{j+1} = 0|\tau_0 = 2). \end{aligned}\tag{B3}$$

Now we use the matrix algebra (3), and the relation $DE = D + E$. Henceforth, we can write:

$$\begin{aligned}\boldsymbol{E}(\tau_j = 1, \tau_{j+1} = 0|\tau_0 = 2) &= \sum_{p=0}^{N-1}\frac{\langle W|G_{j-1,p}DEG_{L-j-1,N-p-1}|V\rangle}{Z_{L,N}} \\ &= \sum_{p=0}^{N-1}\frac{\langle W|G_{j-1,p}DG_{L-j-1,N-p-1}|V\rangle}{Z_{L,N}} + \frac{\langle W|G_{j-1,p}EG_{L-j-1,N-p-1}|V\rangle}{Z_{L,N}} \\ &= \frac{Z_{L-1,N}}{Z_{L,N}}\boldsymbol{E}_{L-1,N}(\tau_j = 1|\tau_0 = 2) + \frac{Z_{L-1,N-1}}{Z_{L,N}}\boldsymbol{E}_{L-1,N-1}(\tau_j = 0|\tau_0 = 2). \end{aligned}\tag{B4}$$

The subscripts $\boldsymbol{E}_{L-1,N}$ mean that we take the averages in a system of size $L-1$ containing N normal particles. These subscripts have been omitted when they were obvious.

We use the relations

$$\sum_{j=1}^{L-1}\boldsymbol{E}_{L-1,N}(\tau_j = 1|\tau_0 = 2) = N$$

and

$$\sum_{j=1}^{L-1}\boldsymbol{E}_{L-1,N-1}(\tau_j = 0|\tau_0 = 2) = L - N$$

to conclude the derivation of formula (16).

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